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Low-Temperature Properties of a Quantum Fermi Gas in Curved Space-Time

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We consider the behavior of an ideal quantum Fermi gas in curved space-time. We obtain and analyze the expressions for the densities of the Helmholtz free energy and grand thermodynamic potential in this case. We find the dependence of chemical potential and Fermi energy on the curvature of space-time and compute the explicit expression for the chemical potential of a Fermi gas at high densities and in the low-temperature approximation.

1. INTRODUCTION

In this work, we develop the methods of local statistical thermodynamics (Kulikov and Pronin, 1993) with application to the ideal fermionic gas in curved space-time. We base our research on the Schwinger–DeWitt proper time formalism (DeWitt, 1965) and a local momentum space method (Bunch and Parker, 1979; Panangaden, 1981) in quantum field theory in an arbitrary curved space-time and also on the imaginary time formalism to introduce the temperature into the model (Dolan and Jackiw, 1974; Weinberg, 1974). To compute thermodynamic potentials of the quantum Fermi gas, we use its connection with finite-temperature Green's functions, which may be found by the local momentum space method. Such an approach allows us to introduce the chemical potential in the model and find its dependence on the curvature of the time-independent background gravitational field. We also show that the Fermi energy will change with the curvature of the space-time.

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2. HELMHOLTZ FREE ENERGY OF THE FERMI GAS IN CURVED SPACE-TIME

We will consider that both fermions and gravity are present. Then the total Lagrangian of the system is

$$L_{\rm tot} = L_{\rm g} + L_{\rm m} \tag{2.1}$$

where

$$L_{\rm g} = \frac{1}{16\pi G}(R - 2\Lambda) + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + \gamma R_{\mu\nu\tau\sigma}R^{\mu\nu\tau\sigma} \tag{2.2}$$

is the gravitational Lagrangian and

$$L_{\rm m} = \frac{i}{2}\bar{\psi}(x)(\gamma^{\mu}(x)D_{\mu} + m)\psi(x) \tag{2.3}$$

is the fermionic matter field Lagrangian. The covariant derivative may be written in the form $D_{\mu} = \partial_{\mu} - \Gamma_{\mu}$, where Γ_{μ} describes the coupling spinors with gravitational field. Matrices $\gamma^{\mu}(x)$ obey anti-commutation relations $\{\gamma^{\mu}(x), \gamma^{\nu}(x)\} = 2g^{\mu\nu}(x)\hat{1}$. The effective action for the matter field in curved space-time (Birrell and Davies, 1982) is

$$S_{\rm eff} = \frac{i}{2}\ln \mathrm{Det}[-G_f] = \int d^4x\,\sqrt{g(x)}L_{\rm eff}(x) \tag{2.4}$$

where

$$L_{\rm eff} = \frac{i}{2}\int_{m^2}^{\infty} dm^2\,\mathrm{tr}\,G_f(x, x') \tag{2.5}$$

is the density of the effective Lagrangian of the matter field. The bispinor Green's function $G_f(x, x')$ is the solution of the equation

$$\left(D^{\mu}D_{\mu} + \frac{1}{4}R - m^2\right)G_f(x, x') = -g(x)^{-1/2}\delta(x - x')\hat{1} \tag{2.6}$$

where R is the scalar of curvature and g is minus the determinant of the background metric $g_{\mu\nu}$. It may be found by the Schwinger–DeWitt procedure (DeWitt, 1965) and has the form

$$G_f(x, x') = i\Delta^{1/2}(x, x')\int_0^{\infty} i\,ds\,(4\pi is)^{-2}$$
$$\times \exp\left(-ism^2 - \frac{\sigma}{2is}\right)F(x, x'; is) \tag{2.7}$$

where $\Delta^{1/2}(x, x')$ is the Van Vleck determinant and $\sigma(x, x')$ is half the square of the geodesic distance between points x and x'. The function $F(x, x'; is)$ is the series

$$F(x, x'; is) = \sum_{0}^{\infty} \hat{\alpha}_j(x, x')(is)^j \tag{2.8}$$

where the coefficients $\hat{\alpha}_j(x, x')$ describe the non-Euclidean space-time structure. In the limit $x = x'$ they are $\hat{\alpha}_j(x, x) = \hat{\alpha}_j(R)$:

$$\hat{\alpha}_0(R) = \hat{1}, \qquad \hat{\alpha}_1(R) = \frac{1}{12} R \cdot \hat{1}$$

$$\hat{\alpha}_2(R) = \left(-\frac{1}{120} R_{;\mu}{}^{\mu} + \frac{1}{288} R^2 - \frac{1}{180} R_{\mu\nu}R^{\mu\nu} + \frac{1}{180} R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\right) \cdot \hat{1}$$
$$+ \frac{1}{48} G_{[\alpha\beta]} G_{[\gamma\delta]} R^{\alpha\beta\lambda\xi} R^{\gamma\delta}{}_{\lambda\xi} \tag{2.9}$$

with $G_{[\alpha\beta]} = \frac{1}{4}[\gamma_\alpha, \gamma_\beta]$.

To introduce temperature, we can assume that $V_{4)}$ is a static manifold with topology $S^1 \times M_{3)}$, where $M_{3)}$ is the spatial three-dimensional manifold, while the time coordinate expands to the interval $[0, -i\beta]$ and β is identified with the inverse temperature (Denardo and Spalucci, 1983). In this case, the Green's function of the fermionic field satisfies the antiperiodic boundary conditions. It leads to discrete momentum $k^0 = i\pi T(2n + 1)$ with $n = 0, \pm 1, \pm 2, \ldots$ in the momentum space. At finite temperature the expression for the Green's function (2.7) is separated into two parts

$$G_f(x, x) \overset{\beta}{=} G_\infty(x, x) + G_\beta(x) \tag{2.10}$$

where $G_\infty(x, x)$ is (2.7), in the limit of coincidence $x = x'$, and

$$G_\beta(x) = \frac{i}{(4\pi)^2} \sum_{j=0}^{\infty} \hat{\alpha}_j[R(x)]$$
$$\times \int_0^\infty i\, ds\, (is)^{j-2} \sum_{n=1}^{\infty} (-1)^n \exp\left(-ism^2 - \frac{n^2\beta^2}{4is}\right) \tag{2.11}$$

is the finite-temperature contribution in the Green's function. Inserting (2.10) into (2.5), we find the finite-temperature density of the effective Lagrangian in the form

$$L_\beta(R) = L_\infty(R) - f(\beta, R) \tag{2.12}$$

The first contribution in the expression (2.12),

$$L_\infty(R) = -\frac{1}{2}(4\pi)^{-n/2}\sum_{j=0}^{\infty}\mathrm{tr}\,\hat{\alpha}_j(R)$$
$$\times\int_0^\infty i\,ds\,(is)^{j-n/2-1}\exp(-ism^2) \qquad (2.13)$$

is temperature independent and divergent at $n = 4$, and the second one may be written in the following way:

$$f(\beta, R) = -\frac{i}{2}\,\mathrm{tr}\int_{m^2}^{\infty} dm^2\, G_\beta(x) = \sum_{j=0}^{3}\alpha_j(R)f_j(\beta m) \qquad (2.14)$$

where $\alpha_j(R) = (1/2s)\,\mathrm{tr}\,\hat{\alpha}_j(R)$ and $2s$ is the dimension of the γ matrices. The coefficients $f_j(\beta m)$ are

$$f_0(\beta m) = \frac{m^2\cdot 2s}{2\pi^2\beta^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}K_2(\beta mn) \qquad (2.15)$$

$$f_1(\beta m) = \frac{m\cdot 2s}{4\pi^2\beta}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}K_1(\beta mn) \qquad (2.16)$$

$$f_2(\beta m) = \frac{2s}{8\pi^2}\sum_{n=1}^{\infty}(-1)^n K_0(\beta mn) \qquad (2.17)$$

and the modified Bessel functions $K_\nu(x)$ are determined by (A.8).

The infinite, temperature-independent contribution $L_\infty(R)$ in the Lagrangian may be canceled by a renormalization procedure of the gravitational Lagrangian (2.2) in the following manner: $\tilde{L}_g(R) = L_g(R) + L_\infty(R)$. This leads to a renormalization of the parameters of the Lagrangian L_g.

The finite, temperature-dependent contribution $f(\beta, R)$ represents the density of the Helmholtz free energy in curved space-time. Using an integral representation for the series of modified Bessel functions (A.10), (A.12), and (A.14), one can write it as a series

$$f(\beta, R) = f_0(\beta m) + \alpha_1(R)f_1(\beta m) + \alpha_2(R)f_2(\beta m) + \cdots \qquad (2.18)$$

where the first term is the standard form of the Helmholtz free energy in Euclidean space

$$f_0(\beta m) = -\frac{2s}{\beta}\int\frac{d^3k}{(2\pi)^3}\ln(1 + e^{-\beta\epsilon}) \qquad (2.19)$$

with particle energy $\epsilon = (\mathbf{k}^2 + m^2)^{1/2}$. The factor $2s = 4$ reflects the existence of the four degrees of freedom present in the fermion field: particles and antiparticles, spin up, and spin down. The following terms are geometrical corrections of the Riemann space-time structure with respect to the Euclidean one with temperature coefficients in the form

$$f_j(\beta m) = -\frac{2s}{\beta} \int \frac{d^3k}{(2\pi)^3} \left(-\frac{\partial}{\partial m^2}\right)^j \ln(1 + e^{-\beta\epsilon}) \tag{2.20}$$

The method developed above does not allow us to compute the density of the grand thermodynamic potential; therefore, in the following calculations we will use the local momentum space formalism as the most convenient for the construction of local thermodynamics.

3. GRAND THERMODYNAMIC POTENTIAL AND LOW-TEMPERATURE PROPERTIES OF FERMI GAS

An equivalent to the Schwinger–DeWitt representation, the momentum space representation of the bispinor $G_f(x, x')$ (Bunch and Parker, 1979), may be written as

$$G_f(x, x') = G_f(x, y) = g^{-1/4}(y) \sum_{j=0}^{2} \hat{\alpha}_j(x, y)\left(-\frac{\partial}{\partial m^2}\right)^j G_0(y) \tag{3.1}$$

where

$$G_0(y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{iky}}{k^2 + m^2} \tag{3.2}$$

The Van Vleck determinant of (2.7) coincides with $g^{-1/4}(y)$, and $\{y^\mu\}$ are normal Euclidean coordinates in tangential space of the point x of manifold $V_{4)}$ in the origin with x' (Petrov, 1969).

To introduce temperature, we will extend the time y^0 coordinate of the tangential space $\{y^\mu\}$ to the imaginary interval $[0, -i\beta]$ and will consider the fermionic field to be antiperiodic on that interval. Then, in the imaginary time formalism,

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m^2} \xrightarrow{\beta} \frac{i}{\beta} \int \frac{d^3k}{(2\pi)^3} \sum_{n=-\infty}^{\infty} \frac{1}{-\omega_n^2 + \epsilon^2} \tag{3.3}$$

where $\omega_n = i\pi T(2n + 1)$, $n = 0, \pm 1, \pm 2, \ldots$, are Matsubara frequencies (Matsubara, 1955). To take into consideration the chemical potential we will make a shift of the frequencies $\omega_n = \omega_n + \mu$ (Morley, 1978; Kapusta, 1979). Making the summation in (3.3), we will find $G_0(y)$ in the limit of coincidence $x = x'$ in the form

$$\lim_{x\to x'} G_0(y) \stackrel{\beta}{=} \frac{i}{\beta} \int \frac{d^3k}{(2\pi)^3} \sum_{n=-\infty}^{\infty} \frac{1}{-(\omega_n + \mu)^2 + \epsilon^2}$$

$$= \frac{i}{\beta} \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{1}{2\epsilon} \sum_{n=-\infty}^{\infty} \left(\frac{\epsilon - \mu}{(\epsilon - \mu)^2 - \omega_n^2} + \frac{\epsilon + \mu}{(\epsilon + \mu)^2 - \omega_n^2} \right) \right\}$$

$$= \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{i}{2\epsilon} \left(\frac{1}{2} - \frac{1}{\exp \beta(\epsilon - \mu) + 1} \right) \right.$$

$$\left. + \frac{i}{2\epsilon} \left(\frac{1}{2} - \frac{1}{\exp \beta(\epsilon + \mu) + 1} \right) \right\}$$

$$= \lim_{x\to x'} [G_\beta^+(y) + G_\beta^-(y)] \tag{3.4}$$

This equation describes temperature contributions for particles (μ) and antiparticles ($-\mu$) separately. One can find the expression for the finite-temperature contribution in the Green's function for fermions with nonzero chemical potential,

$$G_\beta(x, \mu) = \int \frac{d^3k}{(2\pi)^3} \sum_{j=0}^{2} \hat{\alpha}_j(R) \left(\frac{\partial}{\partial m^2} \right)^j (1 + ze^{-\beta\epsilon})^{-1} \tag{3.5}$$

Then the density of the grand thermodynamic potential for fermions may be written as

$$\omega(\beta, \mu, R) = -\frac{i}{2} \operatorname{tr} \int_{m^2}^{\infty} dm^2\, G_\beta(x, \mu) = \sum_{j=0}^{2} \alpha_j(R) f_j(\beta m; z) \tag{3.6}$$

where

$$f_0(\beta m; z) = -\frac{s}{\beta} \int \frac{d^3k}{(2\pi)^3} \ln(1 + ze^{-\beta\epsilon}) \tag{3.7}$$

and

$$f_j(\beta m; z) = \left(-\frac{\partial}{\partial m^2} \right)^j f_0(\beta m; z) \tag{3.8}$$

where $z = e^{\beta\mu}$ is a fugacity and the factor $s = 2$ (spin up, down). The coincidence of these two methods for calculation of the densities of thermodynamic potentials is obvious for $\mu = 0$.

Using the equations for thermodynamic potentials, one can obtain some interesting properties of an ideal Fermi gas in an external gravitational field:

1. The Fermi distribution function of the gas in a gravitational field may be found from the expression $n_{\mathbf{k}} = -\partial\omega_{\mathbf{k}}(\beta, \mu, R)/\partial_\mu$ for occupation numbers with momentum **k**. It has the form

$$n_{\mathbf{k}} = \frac{1}{z^{-1}e^{\beta\epsilon_{\mathbf{k}}} + 1} F(\beta, R) \tag{3.9}$$

where the function $F(\beta, R)$ is described by the formula

$$F(\beta, R) = 1 + \alpha_1(R)\frac{\beta}{2\epsilon_{\mathbf{k}}}[1 - (1 + ze^{-\beta\epsilon_{\mathbf{k}}})^{-1}] + \cdots \tag{3.10}$$

and depends on the curvature, temperature, and energy of the fermion.

2. We can estimate the dependence of the chemical potential on the curvature of space-time in the nonrelativistic approximation. Let $\epsilon_{\mathbf{k}} = \mathbf{k}^2/2m$; then from $n = -\partial\omega(\beta, \mu, R)/\partial\mu$ we find the equation

$$\frac{n\lambda^3(T)}{s} = f_{3/2}(z, R) \tag{3.11}$$

where $\lambda = (2\pi/mT)^{1/2}$ is the thermal wavelength of the particle, and

$$f_{3/2}(z, R) = f_{3/2}(z)\left[\alpha_0 - \frac{3}{4}\frac{\alpha_1(R)}{m^2} - \frac{3}{16}\frac{\alpha_2(R^2)}{m^4} - \cdots\right] \tag{3.12}$$

is some function with respect to z, and n is the density of the Fermi gas. The function $f_{3/2}(z)$ is

$$f_{3/2}(z) = \sum_{n=1}^{\infty}\frac{z^n}{n^{3/2}}(-1)^{n+1} = \frac{4}{\sqrt{\pi}}\int_0^\infty dx\,\frac{x^2}{z^{-1}\exp(x^2) + 1} \tag{3.13}$$

Equation (3.11) may be solved with graphical methods. As can be seen in Fig. 1, the fugacity (chemical potential) depends on the curvature R of the space-time.

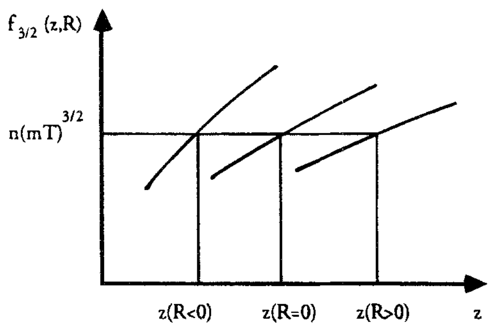

Fig. 1. The dependence of the fugacity on the curvature of space-time for fixed temperature and density.

3. The explicit expression for the chemical potential with respect to low temperatures and high densities ($n\lambda^3 >> 1$), where the quantum effects are essential, may be found by the calculation of (3.11) with the following representation of the function $f_{3/2}(z)$ for large z (Huang, 1963):

$$f_{3/2}(z) = \frac{4}{3\sqrt{\pi}}\left[(\ln z)^{3/2} + \frac{\pi^2}{8}(\ln z)^{-1/2} + \cdots\right] + O(z^{-1}) \qquad (3.14)$$

Inserting (3.12) into (3.11) and taking into account only the first term in (3.14), we find the Fermi energy of a gas of fermions in curved space-time

$$\left(\frac{3n\sqrt{\pi}}{4s}\right)^{2/3}\lambda^2\left[1 + \frac{1}{24}\frac{R}{m^2} + \cdots\right] = \beta\epsilon_F(R) \qquad (3.15)$$

or

$$\epsilon_F(R) = \epsilon_F^{(0)}\left[1 + \frac{1}{24}\frac{R}{m^2} + \cdots\right] \qquad (3.16)$$

where

$$\epsilon_F^{(0)} = \left(\frac{6\pi^2 n}{s}\right)^{2/3}\left(\frac{1}{2m}\right)$$

is the Fermi energy in Euclidean space. Taking into account the second term in (3.14), we get the expression for the chemical potential

$$\mu(T, R) = \epsilon_F(R)\left\{1 - \frac{\pi^2}{12}\left(\frac{T}{\epsilon_F(R)}\right)^2 + \cdots\right\}$$

$$= \epsilon_F^{(0)}\left\{1 + \frac{1}{24}\frac{R}{m^2} - \frac{\pi^2}{12}\left(\frac{T}{\epsilon_F^{(0)}}\right)^2 + \cdots\right\}$$

or

$$\mu(T, R) = \mu^{(0)}(T) + \frac{1}{24}\frac{R}{m^2}\epsilon_F^{(0)} + \cdots \qquad (3.17)$$

which describes the explicit dependence of the chemical potential of the fermionic gas on temperature T and the curvature R of space-time.

4. CONCLUSION

We connected the method of calculation of thermodynamic potentials of a quantum Fermi gas as local thermodynamic objects in a curved space-time with finite-temperature computations of the Green's function of fermions by means of the local momentum space formalism. This allowed us to find the change in the Fermi energy for a Fermi gas in the external gravitational field (3.16) and get the linear dependence of the chemical potential of the gas (3.17) on the curvature of the space-time. In our model the Fermi distribution function depends also on the characteristics of the space-time and expression (3.9) shows its nonthermal character. The temperature of the quantum Fermi gas is a local thermodynamic characteristic of the system and depends on the selected point of the space-time manifold.

APPENDIX. INTEGRAL REPRESENTATIONS OF MODIFIED BESSEL FUNCTIONS

Integral representations for the series of modified Bessel functions (2.15)–(2.17) can be calculated from the following summation formula:

$$\sum_{n=-\infty}^{\infty}\left[y^2+\left(\frac{n+1}{2}\right)^2\right]^{-1}=\frac{\pi}{y}\tanh(\pi y) \tag{A.1}$$

In the proper time representation the left side of equation (A.1) can be written in the form

$$\sum_{n=-\infty}^{\infty}\left[y^2+\left(\frac{n+1}{2}\right)^2\right]^{-1} = \int_0^{\infty} d\alpha\, \exp(-\alpha y^2) \sum_{n=-\infty}^{\infty} \exp\left[-\alpha\left(n+\frac{1}{2}\right)^2\right] \tag{A.2}$$

Taking into account the equation

$$\sum_{n=-\infty}^{\infty} \exp\{-\alpha(n-z)^2\} = \sum_{n=-\infty}^{\infty}\left(\frac{\pi}{\alpha}\right)^{1/2}\exp\left(-\frac{\pi^2}{\alpha}n^2-2\pi i z n\right) \tag{A.3}$$

we can write (A.2) as

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\left[y^2+\left(\frac{n+1}{2}\right)^2\right]^{-1} &= \sum_{n=-\infty}^{\infty}(-1)^n\int_0^{\infty} d\alpha\left(\frac{\pi}{\alpha}\right)^{1/2}\exp\left(-\alpha y^2-\frac{\pi^2}{\alpha}n^2\right)\\ &= \frac{\pi}{y}+2\sum_{n=1}^{\infty}(-1)^n\int_0^{\infty} d\alpha\left(\frac{\pi}{\alpha}\right)^{1/2}\exp\left(-\alpha y^2-\frac{\pi^2}{\alpha}n^2\right)\end{aligned} \tag{A.4}$$

The right side of (A.1) is

$$\frac{\pi}{y}\tanh(\pi y) = \frac{\pi}{y} - \frac{2\pi}{y(e^{2\pi y}+1)} \tag{A.5}$$

Therefore, we get from (A.4) and (A.5) the following useful equation:

$$\frac{1}{z(e^z+1)} = -\frac{1}{2\pi^2}\sum_{n=1}^{\infty}(-1)^n \int_0^{\infty} d\alpha \left(\frac{\pi}{\alpha}\right)^{1/2} \times \exp\left(-\alpha\frac{z^2}{4\pi^2} - \frac{\pi^2}{\alpha}n^2\right) \tag{A.6}$$

Moreover, we may consider that $z^2 = g_a(x^2)$ is a function of variable $x \in R^3$ with a parameter a, namely $z^2 = x^2 + a^2$. Making the integration with respect to x, one gets

$$\int \frac{d^3x}{(2\pi)^3}((x^2+a^2)^{1/2}\{\exp[(x^2+a^2)^{1/2}]+1\})^{-1} = -\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n \int_0^{\infty} d\alpha\, \alpha^{-2}\exp\left(-\alpha\frac{a^2}{4\pi^2} - \frac{\pi^2}{\alpha}n^2\right) \tag{A.7}$$

The modified Bessel function may be written as

$$\int_0^{\infty} d\alpha\cdot\alpha^{\nu-1}\exp\left(-\gamma\alpha - \frac{\delta}{\alpha}\right) = 2\left(\frac{\delta}{\gamma}\right)^{\nu/2} K_\nu(2(\delta\gamma)^{1/2}) \tag{A.8}$$

Then (A.7) becomes

$$\int \frac{d^3x}{(2\pi)^3}((x^2+a^2)^{1/2}\{\exp[(x^2+a^2)^{1/2}]+1\})^{-1} = -\frac{1}{2}\left(\frac{a}{\pi^2}\right)\sum_{n=1}^{\infty}\frac{(-1)^n}{n}K_1(an) \tag{A.9}$$

Scaling x and a with a parameter β as $(x, a) = \beta(k, m)$, we write (A.9) in the form

$$\int \frac{d^3k}{(2\pi)^3}\frac{2}{\epsilon[\exp(\beta\epsilon)+1]} = -\frac{m}{\beta\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}K_1(\beta mn) \tag{A.10}$$

where $\epsilon = (\mathbf{k}^2 + m^2)^{1/2}$. Differentiating (A.7) with respect to the parameter a, we get

$$\int \frac{d^3x}{(2\pi)^3} \left(-\frac{\partial}{\partial a^2}\right)((x^2 + a^2)^{1/2}\{\exp[(x^2 + a^2)^{1/2}] + 1\})^{-1}$$

$$= -\frac{1}{4\pi^2} \sum_{n=1}^{\infty} (-1)^n K_0(an) \tag{A.11}$$

or, in the new variables,

$$\int \frac{d^3k}{(2\pi)^3} \left(\frac{\partial}{\partial m^2}\right) \frac{1}{\epsilon[\exp(\beta\epsilon) + 1]} = \frac{1}{4\pi^2} \sum_{n=1}^{\infty} (-1)^n K_0(\beta mn) \tag{A.12}$$

Making the integration in (A.7) with respect to parameter a and using the equation

$$\int_{a^2}^{\infty} da^2 \, ((x^2 + a^2)^{1/2}\{\exp[(x^2 + a^2)^{1/2}] + 1\})^{-1}$$

$$= 2 \ln\{1 + \exp[-(x^2 + a^2)^{1/2}]\} \tag{A.13}$$

we find in the new variables the equation

$$-\frac{1}{\beta} \int \frac{d^3k}{(2\pi)^3} \ln[1 + \exp(-\beta\epsilon)] \tag{A.14}$$

$$= \frac{m^2}{2(\beta\pi)^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} K_2(\beta mn)$$

The high-temperature asymptotes ($\beta m << 1$) of (A.9), (A.11), and (A.14),

$$\frac{2m^2}{(\beta\pi)^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} K_2(\beta mn) = -\frac{7\pi^2}{180\beta^4} + \frac{m^2}{12\beta^2} + \frac{1}{8} m^4 \left(\ln \frac{\beta m}{4\pi} + \gamma - \frac{3}{4}\right)$$

$$-\frac{m}{\beta\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} K_1(\beta mn) = \frac{1}{12\beta^2} + \frac{1}{4} m^2 \left(\ln\left(\frac{\beta m}{4\pi}\right) + \gamma - \frac{1}{2}\right)$$

$$\frac{1}{2\pi^2} \sum_{n=1}^{\infty} (-1)^n K_0(\beta mn) = \frac{1}{4} \left(\ln\left(\frac{\beta m}{4\pi}\right) + \gamma\right) \tag{A.15}$$

are useful for some thermodynamic calculations.

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REFERENCES

Birrell, N. D., and Davies, P. S. W. (1982). *Quantum Fields in Curved Space-Time,* Cambridge University Press, Cambridge.

Bunch, T., and Parker, L. (1979). *Physical Review D,* **20**, 2490–2510.

Denardo, G., and Spalucci, E. (1983). *Physics Letters,* **130B**, 43–46.

DeWitt, B. S. (1965). *Dynamical Theory of Groups and Fields,* Gordon and Breach, New York.

Dolan, L., and Jackiw, R. (1974). *Physical Review D,* **9**, 3320–3341.

Huang, K. (1963). *Statistical Mechanics,* Wiley, New York.

Kapusta, J. (1979). *Nuclear Physics B,* **148**, 461–498.

Kislinger, M. B., and Morley, P. D. (1976). *Physical Review D,* **13**, 2765–2770, 2771–2777.

Kulikov, I., and Pronin, P. (1993). *International Journal of Theoretical Physics,* **32**, 1261–1273.

Matsubara, T. (1955). *Progress of Theoretical Physics,* **14**, 351–378.

Morley, P. D. (1978). *Physical Review D,* **17**, 598–610.

Panangaden, P. (1981). *Physical Review D,* **23**, 1735–1746.

Petrov, A. Z. (1969). *Einstein Spaces,* Pergamon, Oxford.

Weinberg, S. (1974). *Physical Review D,* **9**, 3357–3378.